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On The Reflexivity of Direct Sum of Reflexive Operators

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ABSTRACT

Sarason did pioneer work on reflexive operator and reflexivity of normal operators, however, he did not used the word reflexive but his results are equivalent to say that every normal operator is reflexive. The word reflexive was suggested by HALMOS and first appeared in H. Rajdavi and P. Rosenthals book `Invariant Subspaces' in 1973.This line of research was continued by Deddens who showed that every isometry in$\mathfrak{B}$ (H) is reflexive. R. Wogen has proved that `every quasi-normal operator is reflexive'. These results of Deddens, Sarason, Wogen are particular cases of theorem of Olin and Thomson which says that all sub-normal operators are reflexive. In other direction, Deddens and Fillmore characterized these operators acting on a finite dimensional space are reflexive. J. B. Conway and Dudziak generalized the result of reflexivity of normal, quasi-normal, sub-normal operators by proving the reflexivity of Von Neumann operators. In this paper we shall discuss one of the question that have been posed by Deddens whether the direct sum A⊕B of Reflexive operators A and B acting on Hilbert space $\mathcal{H}$ and $\mathcal{K}$ respectively is necessarily reflexive. The answer is known to be yes in many cases ,in which additional hypotheses were placed on one or both summand. First we shall discuss the operators for which the direct sum of Reflexive operators is Reflexive. After this we shall discuss the operators on which direct sum of Reflexive operators fails to be reflexive. In general, we can say that direct sum of Reflexive operators is not reflexive. We further modified the result by showing that if A and B are Reflexive operators A⊗B is not Reflexive.

Keywords: Reflexive operators, Normal operators, Subnormal operators, Direct sum of operators.

INTRODUCTION

A bounded linear operator T on a complex separable Hilbert space $\mathcal{H}$ is reflexive if Alg T = Alg LatT, where Alg LatT and AlgT denote respectively the weakly closed algebra of operators which leave invariant every invariant sub-space of T and the weakly closed algebra generated by T and I.

1.1 Theorem: Let A_1 and A_2 be reflexive operators on a Hilbert space $\mathcal{H}$, if A_1 is algebraic then $A_1 \oplus A_2$ is reflexive.

1.2 Corollary: Let A and B be reflexive algebraic operators, then A⊕B is reflexive

1.3Definition:A contraction T on a Hilbert space $\mathcal{H}$ is of class C_0 if T is c.n.u and for some function u in $\mathcal{H}$, u(T)=0.if u=$u_i u_e$ is the canonical factorization of u in to its inner part u_i and outer part u_e, then u(T)=0 if and only if u_i(T)=o.Also $\{u \in \mathcal{H}^\infty : \text{u is inner and } u(T) = 0\}$ has a greatest common diviser m and m(T)=0.This function m is called the minimal function of T.

1.4 Theorem[6]**:** If T_1 and T_2 are contraction of class C_O with minimal function m_1 and m_2 then the following statements are equivalent:

(a): m_1 and m_2 have no common divisor other then 1.

(b):$Alg(T_1 \oplus T_2)=AlgT_1 \oplus AlgT_2$

(c):$Lat(T_1 \oplus T_2)=LatT_1 \oplus LatT_2$

(d):$AlgLat(T_1 \oplus T_2)=AlgLatT_1 \oplus AlgLatT_2$

1.5 Corollary: If $T_1$and T_2 are contractions of class C_0 and if their minimal function have no common divisor other then 1, then if T_1 and T_2 are reflexive

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Proof: By the hypothesis $Alg(T_1 \oplus T_2)=AlgT_1 \oplus AlgT_2$ and $AlgLat(T_1 \oplus T_2)=AlgLatT_1 \oplus AlgLatT_2$

From theorem1.4 ,The result follows immediately . This corollary shows that if T_1 and T_2 are reflexive contractions of class C_0,then $T_1 \oplus T_2$ IS reflexive.

1.6 Now we shall discuss the non reflexivity of the direct sum of reflexive operators.So that in general we can say that direct sum of two reflexive operators need not be reflexive.

Let $\mathfrak{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathfrak{B}_1 = \mathfrak{B}_1(\mathcal{H})$ denote the ideal of trace class operators on $\mathcal{H}$, $\mathfrak{B}(\mathcal{H})$ is the dual of $\mathfrak{B}_1(\mathcal{H})$.Let $\mathcal{A}(T)$ denote the weak* closure of polynomial in T and I.

Let $T \in \mathfrak{B}(\mathcal{H})$

I. Does Alg T $=\{T\}' \cap$ Alg Lat T ? [4,page197]

II. Is $T \oplus T$ reflexive. [5]

III. Does Alg T $=\mathcal{A}(T)$.In addition [3] [7]

IV. If T_1 and T_2 are reflexive, is $T_1 \oplus T_2$ reflexive? We also add one more question here.

V. If A and B are reflexive, Is $A \otimes B$ reflexive?.

Note that the first two questions are related. $T \oplus T$ is reflexive whenever

Alg T $=\{T\}' \cap$ Alg Lat T . An example of a reflexive operators T is also given for which T^2 is not reflexive.

1.7: Definition: A sub space $\mathcal{L}=\{B \in \mathfrak{B}(\mathcal{H}):Bx \in [\mathcal{L}x] \forall\ x \in \mathcal{H}\}$.For algebra $\mathcal{A}$ containing I. This is equivalent to say $\mathcal{A}$ =Alg Lat $\mathcal{A}$ .Further $\mathcal{L}$ is n-reflexive if $\mathcal{L}^{(n)}$ the n-fold inflation of $\mathcal{L}$ is reflexive. Also an operator T is n-reflexive if W(T) is n-reflexive.

1.8: Definition: The annihilator of $\mathcal{L}$, denoted by $\mathcal{L}_\perp$ is the set

$\mathcal{L}_\perp=\{U \in \mathfrak{B}_1:tr(SU)=o \forall\ S \in \mathcal{L}\}$ where $\mathfrak{B}_1$ is the set of trace class operators on $\mathfrak{B}(\mathcal{H})$.

1.9: Proposition [9]:A sub space $\mathfrak{B}(\mathcal{H})$ is n-reflexive if and only if $\mathcal{L}_\perp \cap F_n$ total in $\mathcal{L}_\perp$,where $F_n=F_n(\mathcal{H})$ denotes the set of operators in $\mathfrak{B}(\mathcal{H})$ of rank $\leq$n.

1.10: Lemma [9]: If Alg T$=\{T\}' \cap$ Alg Lat T then T is n-reflexive for every n≥2.

Remark: For any $\mathcal{A} \subset \mathfrak{B}(K \oplus \mathcal{H})$,let $\mathcal{A}^{\sim}=\{A-A_{1,\infty}:A \in \mathcal{A}\}$

1.11: Lemma [9]: $\{Alg\ T\}^{\sim}=(\{T\}' \cap Alg\ Lat\ T)^{\sim}= \{T\}' \cap (Alg\ Lat\ T)^{\sim}$

1.12:Lemma[9]:$(\{T\}' \cap AlgLatT)_{1,\infty}=(AlgLatT)_{1,\infty}=R(\mathcal{L})_{1,\infty}$,

Where $R(\mathcal{L})=\{R \in \mathfrak{B}(\mathcal{H}):Rx \in [\mathcal{L}x] \forall\ x \in \mathcal{H}\}$

1.13:Lemma[2]: If n≥2,then $Alg(T^{(n)})^{\sim}=Alg\ Lat(T^{(n)})^{\sim}$ and $Alg(T^{(n)})^{\sim}$is a reflexive sub space.

1.14: Proposition: For T as in basic construction [9] Alg T$=\{T\}' \cap$ Alg Lat T if and only if $\mathcal{L}$ is reflexive Sub space.

Proof: Using the Lemmas 1.11 and 1.12 we have $AlgT=(AlgT)^{\sim}+[\mathcal{L}]_{1,\infty}$and $\{T\}' \cap$ Alg Lat $T=\{T\}' \cap (AlgLatT)^{\sim}+(AlgLatT)_{1,\infty}= (AlgT)^{\sim}+[R(\mathcal{L})]_{1,\infty}$ but $R(\mathcal{L})=\mathcal{L}$ precisely when $\mathcal{L}$ is reflexive.

1.15: Proposition: For T as in basic construction [9] and n≥2, T is reflexive if and only if $\mathcal{L}$ is n-reflexive

Proof:We have $Alg(T^{(n)})=Alg(T^{(n)})^{\sim}+[\mathcal{L}^{(n)}]_{1,\infty}$ and $AlgLat(T^{(n)})+AlgLat(T^{(n)})^{\sim}+[R(\mathcal{L}^{(n)})]_{1,\infty}$ Lemma 1.13 shows that $Alg(T^{(n)})^{\sim}=(AlgLatT^{(n)})^{\sim}$.

So that $T^{(n)}$ is reflexive if and only if $\mathcal{L}^{(n)}= R(\mathcal{L}^{(n)})$ and if and only if $\mathcal{L}^{(n)}$ is reflexive.

1.16: Example [2]: If $1<n<\infty$, there is a reflexive operator S so that S^2 is not n-reflexive.

Proof: Supose that $\mathcal{L}$ is a wot closed subspace of $\mathfrak{B}(\mathcal{H})$ which is not 2n-reflexive. We can construct an operator T as in basic construction[9] so that $(AlgT)_{1,\infty}= [\mathfrak{B}(\mathcal{H})]_{1,\infty}$ While $Alg(T^2)=[\mathcal{L}^{(n)}]_{1,\infty}$.The idea is that to choose the even entries in the column matrix Q to have wot. Span , while the odd entries are chosen to span $\mathfrak{B}(\mathcal{H})$.

Now let $S=T^2$. By the proposition 1.15 S is reflexive. Since[$Alg(S^2)]_{1,\infty}$ and $\mathcal{L}^{(2)}$ is not n-reflexive.We can see that $S^{(2)}$ is not n-reflexive.

We now give answer to question 4 by constructing a direct sum of reflexive operators which is not reflexive.

1.17 Lemma: Fix $1 \le n < \infty$ and suppose that $\mathcal{M}=\mathcal{C}^{n+1}$.There is a subspace $\mathcal{L}$ of $\mathfrak{B}(\mathcal{M} \oplus \mathcal{M})$ with these properties $\mathcal{M} \oplus 0$ reduces $\mathcal{L}$, the sub spaces $\mathcal{L}_1=\mathcal{L}/\mathcal{M} \oplus 0$ and $\mathcal{L}_2/0\oplus\mathcal{M}$ are reflexive,and $\mathcal{L}$ is not n-reflexive.

Proof: Let $\mathcal{L}=\{S\oplus T:S,T \in \mathfrak{B}(\mathcal{M})$ and trace $(S+T)=0\}$. Then $\mathcal{L}_1= (\mathcal{M})= \mathcal{L}_{2,}$ so that $\mathcal{L}_1$ and $\mathcal{L}_2$ are reflexive.We use proposition 1.9 to show that $\mathcal{L}$ is not n-reflexive by showing that $[S_1 \cap F_n] \neq S_\perp$.

Each $U\in\mathcal{L}_\perp$ has form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A,B,C,D in $\mathfrak{B}(\mathcal{M})$.For every $S\in \mathfrak{B}(\mathcal{M})$ with S=0 .We have $tr(U(S \oplus 0))=trAS=0$.Thus $A=\lambda I$ for some $\lambda \in \mathcal{C}$.In particular,if $U\in S_\perp \cap F_n$,then A=0.Thus $I\oplus I\in S_\perp$but $I\oplus I\notin[S_\perp \cap F_n]$

1.18 Example [9]: If $1\le n < \infty$,then there are reflexive operators S_1 and S_2,So that $S_1\oplus S_2$ is not n-reflexive.

1.19 Defination: Let $\mathcal{H}$ be a complex Hilbert space and let $\mathfrak{B}(\mathcal{H})$ denote the algebra of bounded linear operator on $\mathcal{H}$.For a linear sub-space $\mathcal{L}$ of $\mathfrak{B}(\mathcal{H})$

Ref $\mathcal{L} =\{B\in \mathfrak{B}(\mathcal{H}):B_X \in [\mathcal{L}_X],x\in \mathcal{H}\}$

If $A\in \mathfrak{B}(\mathcal{H})$,then $\mathcal{W}(A)$ will denote the closure in the weak topology of $\mathfrak{B}(\mathcal{H})$of the set P(A)of polynomial in A, and $\mathcal{W}_0(A)$will denote the weakly closed principal ideal generated by A. Thus $\mathcal{W}_0(A)$is the closure in the wot of the linear span of the positive power of A, and it may happen that $\mathcal{W}_0(A)= \mathcal{W}(A)$.So the operator A will be reflexive if

$\mathcal{W}(A)$=AlgLat A =Ref $\mathcal{W}(A)$

1.20 Proposition [1]: Let $A\in \mathfrak{B}(\mathcal{H})$ then:

(1)Lat(A⊕O)splits if and only if I∊ Ref $\mathcal{W}(A)$.

(2) (A⊕O)splits if and only if I ∊ $\mathcal{W}_0(A)$.

1.21 Proposition: Let $\mathcal{H}$and $\mathcal{K}$ be Hilbert space with dim $k\ge1$.Let $A\in \mathfrak{B}(\mathcal{H})$,and 0 denote the zero transformation on $\mathcal{K}$,then

(1) A⊕O is reflexive if and if $\mathcal{W}_0(A)$is reflexive.

(2)If A is reflexive, then A⊕O fails to be reflexive if and only if $I\notin \mathcal{W}_0(A)$ but I∊ Ref $\mathcal{W}_0(A)$.

Proof: Let us suppose that $\mathcal{W}_0(A)$is reflexive.

Let B∊ Ref ((A⊕O))be arbitrary. Since Ref($\mathcal{W}(A) \oplus \mathcal{C}I$)is a reflexive algebra containing A⊕O,it follows that $B=B_1\oplus\lambda I$ for some $B_1\in$ Ref $\mathcal{W}(A)$ and $\lambda\in \mathcal{C}$.We shall show that $B_1- \lambda I\in \mathcal{W}_0(A)$,and hence

Since $\mathcal{W}(A\oplus O)= \mathcal{W}_0(A\oplus O)+ \mathcal{C}(I\oplus I)$ and $\mathcal{W}_0(A\oplus O)= \mathcal{W}_0(A)\oplus O$,This will prove that $B \in\mathcal{W}(A\oplus O)$,as required.

Fix a non zero vector $y\in \mathcal{K}$.Let $x \in \mathcal{H}$ be arbitrary. Since B∊Ref $\mathcal{W}(A\oplus O)$,there exist a sequence of polynomial $\{ p_n\}$ depending on x,such that

$Lim_n(p_n(A\oplus O))(x\oplus y)=(B_1\oplus \lambda I) (x\oplus y)=(B_1x) \oplus(\lambda y)$

Since $p_n(A\oplus O)=p_n(A) \oplus(p_n(O)I)$,

We must have $p_n(0)\to \lambda$ and $p_n(A)x\to B_1(x)$

Let $q_n(0)=p_n-p_n(0)$ then $q_n(0)=0$ and $q_n(A)x\to(B_1- \lambda I)x$.Thus

$(B_1- \lambda I)x\in [\mathcal{W}_0(A)x]$.Since x is arbitrary, this shows that

$B_1- \lambda I\in Ref(\mathcal{W}_0(A))=\mathcal{W}_0(A)$,As needed.So A⊕0 is reflexive.

For the converse let A⊕0 is reflexive.If C∊ Ref($\mathcal{W}_0(A)$),then

$C\oplus 0 \in Ref(\mathcal{W}_0(A))\oplus 0 = Ref(\mathcal{W}_0(A)\oplus 0) = Ref(\mathcal{W}_0(A\oplus 0)) \subseteq Ref(\mathcal{W}(A\oplus 0) = \mathcal{W}(A\oplus 0) = \mathcal{W}(A\oplus 0) + \mathcal{C}(I\oplus I)$.

It follows that $C\in\mathcal{W}_0(A)$ as needed. So $\mathcal{W}_0(A)$ is reflexive.

(2) Follows from (1).Assume that A is reflexive .We have

$\mathcal{W}_0(A) \subseteq Ref(\mathcal{W}_0(A) = Ref\ \mathcal{W}(A) = \mathcal{W}(A) = \mathcal{W}_0(A) + \mathcal{C}I$

From this it is clear that the only way in which $A\oplus 0$, and hence $\mathcal{W}_0(A)$, can fail to be reflexive is if $I\notin \mathcal{W}_0(A)$ but $I\in Ref(\mathcal{W}_0(A))$

1.22 Lemma [1]: $I \in Ref\ (\mathcal{W}_0(T)$

1.23 Lemma [1]: T is a reflexive operator, then $\mathcal{W}(T) = \mathcal{A}(T)$

1.24 Lemma [1]: T is 2-elementory.The relative weak operator topology coincides with the relative * weak operator topology on $\mathcal{W}(T)$

1.25 Lemma: $I\notin \mathcal{W}_0(T)$.

Proof: By the Lemma1.24 $\mathcal{W}_0(T)$ is the weak * closure of the linear span of the idempotent $\{Q_k\}_{k=1}^{\infty}$.Let $h=(h_{ij})$ be the operator defined in term of its coordinate elements by

$h_{kk}=2^{-k}$

$h_{2k-1}=-4^{-k}(\sum_{j=1}^{2k-1} 2^{-j}$

$h_{2k+1,2k}==-4^{-k}(\sum_{j=1}^{2k} 2^{-j}$

For all $k\geq 1$ and all other elements 0, then $h\in \mathfrak{B}_1(\mathcal{H})$.Since it is supported on finitely many (three) diagonal S and each diagonal is absolutely summable.It can be verified that $tr(Q_k h)=0\ \forall\ k\geq 1$. [1]

So $h\in \mathcal{W}_0(T)_{\perp}$. Also (tr h) =1. This shows that $I\notin \mathcal{W}_0(T)$.

1.26 Theorem: T is reflexive but $T\oplus 0$ is not reflexive.

Proof: It was shown in Lemma 1.23, that T is reflexive on the other hand Lemma 1.22 and 1.24 show that $\mathcal{W}_0(T)$ is not reflexive,by the proposition1.21 $T\oplus 0$ fails to be reflexive.

1.27 Theorem: Direct sum of two reflexive operators need not to be reflexive.

Proof: Let T be a reflexive operator and 0 is null operator. Since 0-operator is trivially reflexive and elementary, Theorem 1.26 says that $T\oplus 0$ is not reflexive. This shows that the direct sum of two reflexive operators need not to be reflexive even under the hypothesis that one is elementary and other is 2-elementory.

1.28 Theorem: If A and B are reflexive operators then $A \otimes B$ need not to be reflexive.

Proof: Let A be a reflexive operators and B is a reflexive and rank one projection on a two dimension Hilbert space .Fix an ortho normal basis $\{e_n\}^{\infty}$, for an infinite dimensional Hilbebt space $\mathcal{H}$. View each operator $A \in \mathfrak{B}(\mathcal{H})$ as an infinite matrix $A=(A_{jk})$,$j,k\geq 1$.Let E_{jk} be the unit matrix which has 1 as its (j,k) element and all other elements o. Let $M_n=[e_1, e_2, e_3,............e_n]$and P_n be the orthogonal projection on to M_n.For each $k\geq 1$,let

$Q_{2k-1}=P_{2k-1}+4^k E_{2k,2k-1}$

$Q_{2k}=P_{2k}+4^k E_{2k,2k+1}$

Set $Q_0=0$.Observe that each Q_n is an idempotent of rank n and that range $(Q_n) \subset range(Q_{n+1})$.Also if m<n then $QmQn=QnQm=Q_m$.For $k\geq 1$set $T_k=Q_k-Q_{k-1}$.Each T_k is a rank one idempotent and $T_jT_k=0$if $j\neq k$,we have

$A_1=E_{11}+4E_{21}$

$A_{2k-1}=-4^{k-1}E_{2k-2,2k-1}+E_{2k-1,2h-1}+4^k E_{2k,2k-1}$ for $k \geq 2$.

$A_{2k}=-4^k E_{2k,2k-1}+E_{2k,2h}+4^k E_{2k,2k-1}$ for $k\geq 1$

Thus A_{2k-1} has non zero entries only in the 2k-1 column and A_{2k} has non zero entries only in the 2k row. Since B be a rank one projection on a two dimensional Hilbert space, then $A \otimes B$ is equivalent to T⊕0, which is not reflexive by theorem 1.26.Since equivalence preserves the reflexivity, so A ⊗ B is not reflexive.

CONCLUSION

In the light of above discussion it is clear that direct sum of two reflexive operators is not reflexive. If A and B are two reflexive operators then A ⊗ B is not reflexive.

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