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Local Analytic Solutions of an Iterative Functional Differential Equation Depend on the State Derivative

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ABSTRACT

This paper is concerned with the existence of analytic solutions of a class of iterative functional differential equation of the form $x'(z)=x(az+\frac{b}{x'(z)}),\ z\in C$ .where a, b are two complex numbers. By constructing a convergent power series solution of an auxiliary equation of the form

$$[ag'(z)-\beta g'(\beta z)][g(\beta^2 z)-ag(\beta z)]=\beta[g(\beta z)-ag(z)]^2 g'(\beta z),\ z\in C$$

local analytic solutions for the original equation are obtained. We discuss not only the constant β at resonance, i.e. at a root of the unity, but also those β near resonance (near a root of the unity) under Brjuno condition.

Keywords: Iterative functional differential equation; analytic solution; Diophantine condition; Brjuno condition

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INTRODUCTION

Functional differential equations of the form

$$x'(z)=f(z,x(z-\tau(z))), \tag{1.1}$$

have been lucubrated in [1,2]. However, such equations, when the delay function $\tau(z)$ depends not only on the argument of the unknown function, but also the state or state derivative $\tau(z)=\tau(z,x(z),x'(z))$, have been relatively little researched. In [3, 4], the authors studied the existence of analytic solution of the equations

$$\alpha z+\beta x'(z)=x(az+bx'(z))$$

and

$$x'(z)=x(az+bx'(z))$$

respectively. Taking $f(z,x)=x$, and $\tau(z)=(1-a)z-\frac{b}{x'(z)}$ in (1.1), we deduce the equation of the form

$$x'(z)=x(az+\frac{b}{x'(z)}),\ z\in C \tag{1.2}$$

where a and b are complex numbers, $x(z)$ denotes the unknown complex function. The purpose of this paper is to discuss the existence of analytic solutions to (1.2) in the complex field.

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Since such equations are quite different from the usual differential equations, the standard existence and uniqueness theorems cannot be applied directly. It is therefore of interest to find some or all of their solutions under appropriate conditions.

Let $a \neq 0, b = 0$, Eq.(1.2) changes into the functional differential equation

$$x'(z) = x(az),$$

which has an entire solution of the form(see Elbert[5])

$$x(z) = \sum_{n=0}^{\infty} \frac{a^{\frac{n(n-1)}{2}}}{n!} \eta z^n .$$

A distinctive feature of Eq.(1.2) when $b \neq 0$, is that the argument of the unknown function is dependent on the state derivative $x'(z)$, and this is the case we will emphasize in this paper. Let

$$y(z) = az + \frac{b}{x'(z)}, \tag{1.3}$$

then

$$x'(z) = \frac{b}{y(z) - az}.$$

Then for any number z_0 , we have

$$x(z) = x(z_0) + \int_{z_0}^{z} \frac{b}{y(s) - as} ds , \tag{1.4}$$

and so

$$x(y(z)) = x(z_0) + \int_{z_0}^{y(z)} \frac{b}{y(s) - as} ds .$$

Therefore, in view of Eq.(1.2) and $x'(z) = \frac{b}{y(z) - az}$, we have

$$\frac{b}{y(z) - az} = x(z_0) + \int_{z_0}^{y(z)} \frac{b}{y(s) - as} ds , \tag{1.5}$$

If z_0 is a fixed point of $y(z)$, i.e., $y(z_0) = z_0$, we see that

$$\frac{b}{y(z_0) - az_0} = x(z_0) + \int_{z_0}^{y(z_0)} \frac{b}{y(s) - as} ds ,$$

or

$$x(z_0) = \frac{b}{(1-a) y(z_0)}. \tag{1.6}$$

Furthermore, differentiating both side of (1.5) with respect to z , we obtain

$$\frac{a - y'(z)}{[y(z) - az]^2} = \frac{y'(z)}{y(y(z)) - ay(z)}.$$

that is

$$(a - y'(z))[y(y(z)) - ay(z)] = [y(z) - az]^2 y'(z). \tag{1.7}$$

To find analytic solution of (1.7), as in our previous works [6-13], we first seek an analytic solution $g(z)$ of the auxiliary equation

$$[ag'(z)-\beta g'(\beta z)][g(\beta^2 z)-ag(\beta z)]=\beta[g(\beta z)-ag(z)]^2 g'(\beta z),\ z\in C \tag{1.8}$$

satisfying the initial value condition

$$g(0)=\mu$$

where μ is a complex number.

We will assume that β in Eq.(1.8) satisfies one of the following conditions:

$(H_1)\, 0<|\beta|<1;$

$(H_2)\, \beta=e^{2\pi i\theta}$, where $\theta\in R\setminus Q$ is a Brjuno number [14, 15, i.e. $B(\theta)=\sum_{k=0}^{\infty}\frac{\log q_{k+1}}{q_k}<+\infty$ where $\{p_k/q_k\}$ denotes the sequence of partial fraction of the continued fraction expansion of θ, said to satisfy the Brjuno condition;

$(H_3)\, \beta=e^{2\pi iq/p}$, for some integer $p\in N$, , with $p\geq 2$ and $q\in Z\setminus\{0\}$,and $\alpha\neq e^{2\pi il/k}$, for all $1\leq k\leq p-1$, and $l\in Z\setminus\{0\}$.

We observe that β is inside the unit circle S^1 in the case of (H_1) but on S^1 in the rest cases. More difficulties are encountered for β on S^1, since the small divisor β^n-1 is involved in the latter (2.9). Under Diophantine condition:" $\beta=e^{2\pi i\theta}$, where $\theta\in R\setminus Q$ and there exist constants $\zeta>0$ and $\sigma>0$ such that $|\beta^n-1|\geq\zeta^{-1}n^{-\sigma}$ for all $n\geq 1$", The number $\beta\in S^1$ is "far" from all roots of the unity and was considered in different settings [14]. Since then, we have been striving to give a result of analytic solutions for those β "near" a root of the unity, i.e., neither being roots of the unity nor satisfying the Diophantine condition. The Brjuno condition in (H_2) provides such a change for us. Moreover, we also discuss the so-called the resonance case. i.e.. the case of (H_3) .

ANALYTIC SOLUTION OF THE AUXILIARY EQUATION

Theorem2.1. Suppose (H_1) holds and $a\neq 0,1,\beta$. Then for any $\eta\in C\setminus\{0\}$, Eq.(1.8) has an analytic solution with the form

$$g(z)=\mu+\eta z+\sum_{n=2}^{\infty}b_n z^n, \tag{2.1}$$

in a neighborhood of the origin, where $\mu=\frac{(a-\beta)}{\beta(1-a)}$.

Proof. We seek a solution of Eq.(1.8) in a power series of the form

$$g(z)=\sum_{n=0}^{\infty}b_n z^n, \tag{2.2}$$

where $b_0=\mu$. By substituting (2.2) into (1.8), we get

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}(i+1)(a-\beta^{i+1})\beta^{n-i}(\beta^{n-i}-a)b_{i+1}b_{n-i}\right)z^n$$

$$=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{k=0}^{n-i}(i+1)\beta^{i+1}(\beta^k-a)(\beta^{n-i-k}-a)b_{i+1}b_k b_{n-i-k}\right)z^n$$

Comparing coefficients we obtain

$$(1-\alpha)[\beta(1-\alpha)b_0-(\alpha-\beta)]b_0b_1=0, \tag{2.3}$$

And

$$\begin{aligned}&\frac{\alpha}{\beta}(n+1)(\alpha-\beta)(\beta^n-1)b_{n+1}\\&=\sum_{i=0}^{n-1}(i+1)(\alpha-\beta^{i+1})\beta^{n-i}(\beta^{n-i}-\alpha)b_{i+1}b_{n-i}+\sum_{i=0}^{n-1}\sum_{k=0}^{n-i}(i+1)\beta^{i+1}(\alpha-\beta^k)(\beta^{n-i-k}-\alpha)b_{i+1}b_kb_{n-i-k},\\&n=1,2,\cdots,\end{aligned} \tag{2.4}$$

In view of the definition of μ, we see that $\beta(1-\alpha)b_0-(\alpha-\beta)=0$. So, we can choose b_1 to be η in (2.3). Once b_0 and b_1 are determined, the other terms of the sequence $\{b_n\}_{n=0}^{\infty}$ can be determined successively from (2.4) in a unique manner. Now, we show that the power series (2.1) converges in a neighborhood of the origin.

Since $0<|\beta|<1$, we have

$$\left|\frac{\beta(i+1)\beta^{n-i}(\alpha-\beta^{i+1})(\beta^{n-i}-\alpha)}{\alpha(n+1)(\beta^n-1)(\alpha-\beta)}\right|\le\frac{(1+|\alpha|)^2}{|\alpha||\alpha-\beta||\beta^n-1|}\le M$$

for some positive M. Thus, from (2.4) we obtain

$$|b_{n+1}|\le M\left(\sum_{i=0}^{n-1}|b_{i+1}||b_{n-i}|+\sum_{i=0}^{n-1}\sum_{k=0}^{n-i}|b_{i+1}||b_k||b_{n-i-k}|\right),\quad n=1,2,\cdots, \tag{2.5}$$

If we define a sequence $\{B_n\}_{n=0}^{\infty}$, by $B_0=|\mu|, B_1=|\eta|$, and

$$B_{n+1}=M\left(\sum_{i=0}^{n-1}B_{i+1}B_{n-i}+\sum_{i=0}^{n-1}\sum_{k=0}^{n-i}B_{i+1}B_kB_{n-i-k}\right),\quad n=1,2,\cdots,$$

then in view of (2.5), by induction, we can prove $|b_n|\le B_n, n=1,2,\cdots$. Now, we define the function

$$H(z,\mu,\eta,M)=\sum_{n=0}^{\infty}B_nz^n, \tag{2.6}$$

then

$$\begin{aligned}H^2(z,\mu,\eta,M)&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}B_kB_{n-k}\right)z^n\\&=\left(|\mu|+\sum_{n=0}^{\infty}B_{n+1}z^{n+1}\right)\sum_{n=0}^{\infty}B_nz^n\\&=|\mu|H(z,\mu,\eta,M)+\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}B_{i+1}B_i\right)z^{n+1}\\&=|\mu|H(z,\mu,\eta,M)+|\mu||\eta|z+\sum_{n=1}^{\infty}\left(|\mu|B_{n+1}+\sum_{i=0}^{n-1}B_{i+1}B_i\right)z^{n+1}\\&=2|\mu|H(z,\mu,\eta,M)-|\mu|^2+\sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1}B_{i+1}B_i\right)z^{n+1}\end{aligned}$$

$$H^3(z,\mu,\eta,M)$$
$$=(|\mu|+\sum_{n=0}^{\infty}B_{n+1}z^{n+1})[\sum_{n=0}^{\infty}\sum_{k=0}^{n}B_kB_{n-k})z^n]$$
$$=|\mu|H^2(z,\mu,\eta,M)+\sum_{n=0}^{\infty}(\sum_{i=0}^{n}\sum_{k=0}^{n-i}B_{i+1}B_kB_{n-i-k})z^{n+1}$$
$$=|\mu|H^2(z,\mu,\eta,M)+|\mu|^2|\eta|z+\sum_{n=1}^{\infty}|\mu|^2B_{n+1}z^{n+1}+\sum_{n=1}^{\infty}(\sum_{i=0}^{n-1}\sum_{k=0}^{n-i}B_{i+1}B_kB_{n-i-k})z^{n+1}$$
$$=|\mu|H^2(z,\mu,\eta,M)+|\mu|^2|\eta|z+|\mu|^2(H(z,\mu,\eta,M)-|\mu|-|\eta|z)+\sum_{n=1}^{\infty}(\frac{1}{M}B_{n+1}-\sum_{i=0}^{n-1}B_{i+1}B_{n-i})z^{n+1}$$
$$=(|\mu|-1)H^2(z,\mu,\eta,M)+(|\mu|^2+\frac{1}{M}+2|\mu|)H(z,\mu,\eta,M)-\frac{|\eta|}{M}z-(|\mu|^2+|\mu|^3+\frac{|\mu|}{M}),$$

that is ,

$$H^3(z,\mu,\eta,M)-(|\mu|-1)H^2(z,\mu,\eta,M)-(|\mu|^2+\frac{1}{M}+2|\mu|)H(z,\mu,\eta,M)+\frac{|\eta|}{M}z+(|\mu|^2+|\mu|^3+\frac{|\mu|}{M})=0. \qquad (2.7)$$

Define the function

$$R(z,\varsigma,\mu,\eta,M)=\varsigma^3-(|\mu|-1)\varsigma^2-(|\mu|^2+\frac{1}{M}+2|\mu|)\varsigma+\frac{|\eta|}{M}z+(|\mu|^2+|\mu|^3+\frac{|\mu|}{M}). \qquad (2.8)$$

For (z,ς) from a neighborhood of $(0,|\mu|)$. Since $R(0,|\mu|,\mu,\eta,M)=0$, and

$R'_{\varsigma}(0,|\mu|,\mu,\eta,M)=-\frac{1}{M}$, according to the implicit function theorem, there exists a unique function $\varsigma(z,\mu,\eta,M)$, analytic in a neighborhood of zero, such that

$$\varsigma(0,\mu,\eta,M)=|\mu|,\quad \varsigma'_z(0,\mu,\eta,M)=|\eta|$$

and $R(z,\varsigma,\mu,\eta,M)=0$. By (2.6) and (2.7), we have $H(z,\mu,\eta,M)=\varsigma(z,\mu,\eta,M)$. It follows that the power series (2.6), and hence also (2.1), converges in a neighborhood of the origin. The proof is complete.

Next we devote to the existence of analytic solution of (1.6) under the Brjuno condition. To do this, we first recall briefly the definition of Brjuno numbers and some basic facts. As stated in [15], for a real number θ , we let $[\theta]$ denote its integer part and $\{\theta\}=\theta-[\theta]$ denote its fractional part. Then every irrational number $[\theta]$ has a unique expression of the $Gauss$ ' , continued fraction

$$\theta=a_0+\theta_0=a_0+\frac{1}{a_1+\theta_1}=\cdots,$$

denoted simply by $\theta=[a_0,a_1,\cdots,a_n,\cdots]$, where a_j's and θ_j's are calculated by the algorithm: $(a)\, a_0=[\theta],\ \theta_0=\{\theta\}$, and $(b)\, a_n=[\frac{1}{\theta_{n-1}}],\ \theta_n=\{\frac{1}{\theta_{n-1}}\}$ for all $n\ge 1$. Define the sequences $(p_n)_{n\in N}$ and $(q_n)_{n\in N}$ as follows:

$$q_{-2}=1,\qquad q_{-1}=0,\qquad q_n=a_nq_{n-1}+q_{n-2},$$
$$p_{-2}=0,\qquad p_{-1}=1,\qquad p_n=a_np_{n-1}+p_{n-2}.$$

It is easy to show that $p_n/q_n=[a_0,a_1,\cdots,a_n,\cdots]$. Thus, for every $\theta\in R\setminus Q$, we associate, using its convergence, an arithmetical function $B(\theta)=\sum_{n=0}^{\infty}\frac{\log q_{n+1}}{q_n}$. We say that θ is a Brjuno number or that it satisfies Brjuno condition if $B(\theta) < +\infty$. The Brjuno condition is weaker than the Diophantine condition. For example, if $a_{n+1} \le ce^{a_n}$ for all $n \ge 0$, where $c > 0$ is a constant, then $\theta = [a_0, a_1, \cdots, a_n, \cdots]$. is a Brjuno number but is not a Diophantine condition. So, the case (H_2) contains both Diophantine condition and a part of α "near" resonance. Let $\theta \in R \setminus Q$ and $(q_n)_{n \in N}$ be the sequence of partial denominators of the $Gauss'$, continued fraction for θ. As in [15], let

$$A_k = \left\{ n \ge 0 \,\middle|\, \|n\theta\| \le \frac{1}{8q_k} \right\}, \qquad E_k = \max(q_k, \frac{q_{k+1}}{4}), \qquad \eta_k = \frac{q_k}{E_k}.$$

Let A_k^* be the set of integers $j \ge 0$ such that either $j \in A_k$ or for some j_1 and j_2 in A_k, with $j_1 - j_2 < E_k$, One has $j_1 < j < j_2$ and q_k divides $j - j_1$. For any integer $n \ge 0$, define

$$l_k(n) = \max((1 + \eta_k)\frac{n}{q_k} - 2, (m_n\eta_k + n)\frac{1}{q_k} - 1),$$

where $m_n = \max\{ j \mid 0 \le j \le n, j \in A_k^* \}$. We then define function $h_k : N \to R_+$ as follows:

$$h_k(n) = \begin{cases} \dfrac{m_n + \eta_k n}{q_k} - 1, & if \quad m_n + q_k \in A_k^*, \\ l_k(n), & if \quad m_n + q_k \notin A_k^*. \end{cases}$$

Let $g_k(n) := \max(h_k(n), [\frac{n}{q_k}])$, and define $k(n)$ by the condition $q_{k(n)} \le n \le q_{k(n)+1}$. Clearly, $k(n)$ is non-decreasing. Then we are able to state the following result.

Lemma2.1. ($Davie's$ lemma [16]) Let $K(n) = n\log 2 + \sum_{k=0}^{k(n)} g_k(n)\log(2q_{k+1})$ .Then

(a) there is a universal constant $\gamma > 0$ (independent of n and θ), such that

$$K(n) \le n(\sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k} + \gamma);$$

(b) $K(n_1) + K(n_2) \le K(n_1 + n_2)$ for all n_1 and n_2, and

(c) $-\log|\alpha^n - 1| \le K(n) - K(n-1)$.

Theorem2.2. Suppose (H_2) holds and $a \ne 0,1,\beta$. Then for any $\eta \in C \setminus \{0\}$, Eq.(1.8) has an analytic solution of the form (2.1) in a neighborhood of the origin, where μ is the same number defined in theorem 2.1.

Proof. As in the proof of Theorem2.1, we seek a power series solution of the form (2.1). Set $b_0 = \mu$ and $b_1 = \eta$. Then (2.4) again holds. From (2.4), we have

$$|b_{n+1}| \le \frac{L}{|1 - \beta^n|}(\sum_{i=0}^{n-1}|b_{i+1}||b_{n-i}| + \sum_{i=0}^{n-1}\sum_{k=0}^{n-i}|b_{i+1}||b_k||b_{n-i-k}|, \quad n = 1,2,\cdots, \tag{2.9}$$

where $L = \dfrac{(1+|a|)^2}{|a||a-\beta|}$. To construct a majorant series of (2.1), we consider the implicit functional equation

$$R(z, \varphi, \mu, \eta, L) = 0, \tag{2.10}$$

where R is defined in (2.8). Similarly to the proof of Theorem2.1, using the implicit function theorem we can prove that (2.10) has a unique analytic solution $\varphi(z,\mu,\eta,L)$ in a neighborhood of the origin such that $\varphi(0,\mu,\eta,L)=|\mu|$, and $\varphi'_z(0,\mu,\eta,L)=|\eta|$, Thus

$\varphi(z,\mu,\eta,L)$ in (2.10) can be expanded into a convergent series

$$\varphi(z,\mu,\eta,L)=\sum_{n=0}^{\infty} C_n z^n, \tag{2.11}$$

in a neighborhood of the origin. Replacing (2.11) into (2.10) and comparing coefficients we obtain that $C_0=|\mu|$, $C_1=|\eta|$ and

$$C_{n+1}=L(\sum_{i=0}^{n-1} C_{i+1}C_{n-i}+\sum_{i=0}^{n-1}\sum_{k=0}^{n-i} C_{i+1}C_kC_{n-i-k}, \quad n=1,2,\cdots, \tag{2.12}$$

Note that the series (2.11) converges in a neighborhood of the origin. Hence there is a constant $T>0$ such that

$$C_n \le T^n, n=1,2,\cdots. \tag{2.13}$$

Now, we can deduce, by induction, that $|b_n|\le C_n e^{K(n-1)}$ for $n\ge 1$, where $K:N\to R$ is defined in **Lamma2.1.** In fact $|b_1|=|\eta|=C_1$. For inductive proof we assume that $|b_j|\le C_j e^{K(j-1)}$, $j\le m$. From (2.9) and Lamma 2.1

$$|b_{m+1}|\le \frac{L}{|\beta^m-1|}(\sum_{i=0}^{m-1}|b_{i+1}||b_{m-i}|+\sum_{i=0}^{m-1}\sum_{k=0}^{n-i}|b_{i+1}||b_k||b_{n-i-k}|)$$

$$\le \frac{L}{|\beta^m-1|}(\sum_{i=0}^{m-1}C_{i+1}C_{m-i}e^{K(i)+K(m-i-1)}+\sum_{i=0}^{m-1}\sum_{k=0}^{m-i}C_{i+1}C_kC_{n-i-k}e^{K(i)+K(k-1)+K(n-i-k-1)}).$$

Note that

$$K(i)+K(m-i-1)\le K(m-1)\le \log|\beta^m-1|+K(m).$$

and

$$K(i)+K(k-1)+K(m-i-k-1)\le K(m-2)\le K(m-1)\le \log|\beta^m-1|+K(m).$$

Then

$$|b_{m+1}|\le Le^{K(m)}(\sum_{i=0}^{m-1}C_{i+1}C_{m-i}+\sum_{i=0}^{m-1}\sum_{k=0}^{m-i}C_{i+1}C_kC_{n-i-k})=C_{m+1}e^{K(m)}.$$

as desired. Note that $K(n)\le n(B(\theta)+\gamma)$ for some universal constant $\gamma>0$. Then

$$|b_n|\le T^n e^{(n-1)(B(\theta)+\gamma)},$$

that is

$$\lim_{n\to\infty}\sup(|b_n|)^{\frac{1}{n}}\le \lim_{n\to\infty}\sup(T^n e^{(n-1)(B(\theta)+\gamma)})^{\frac{1}{n}}=Te^{B(\theta)+\gamma}.$$

This implies that the convergence radius of (2.1) is at least $(Te^{B(\theta)+\gamma})^{-1}$. This completes the proof.

In case (H_3) the constant β is not only on the unit circle in C, but also a root of unity. In such a case, the resonant case, both Diophantine condition and Brjuno condition are not satisfied.

Let $\{D_n\}_{n=0}^{\infty}$ be a sequence define by $D_0 = |\mu|$, $D_1 = |\eta|$ and

$$D_{n+1} = \Gamma L(\sum_{i=0}^{n-1} D_{i+1}D_{n-i} + \sum_{i=0}^{n-1}\sum_{k=0}^{n-i} D_{i+1}D_kD_{n-i-k}), \quad n = 1,2,\cdots. \tag{2.14}$$

Where $\Gamma = \max\{1, |\beta^i - 1|^{-1} : i = 1,2,\cdots, p-1\}$ and L is defined in Theorem 2.2.

Theorem2.3. Suppose (H_3) holds and $a \neq 0,1,\beta$, and p is given as above. Let $\{b_n\}_{n=0}^{\infty}$ be determined recursively by $b_0 = \mu$, $b_1 = \eta$ and

$$\frac{a}{\beta}(n+1)(a-\beta)(\beta^n - 1)b_{n+1} = \Theta(n,\beta), \quad n = 1,2,\cdots. \tag{2.15}$$

where

$$\Theta(n,\beta) = \sum_{i=0}^{n-1}(i+1)(a-\beta^{i+1})\beta^{n-i}(\beta^{n-i}-a)b_{i+1}b_{n-i} + \sum_{i=0}^{n-1}\sum_{k=0}^{n-i}(i+1)\beta^{i+1}(a-\beta^k)(\beta^{n-i-k}-a)b_{i+1}b_kb_{n-i-k}.$$

If $\Theta(vp,\beta) = 0$ for all $v = 1,2,\cdots$, then Eq.(1.8) has an analytic solution of the form

$$g(z) = \mu + \eta z + \sum_{n=vp+1, v\in N}\xi_{vp+1}z^n + \sum_{n\neq vp+1, v\in N} b_n z^n, \quad N = \{1,2,3,\cdots\}.$$

in a neighborhood of the origin, where all ξ_{vp+1}'s are arbitrary constants satisfying the inequality $|\xi_{vp+1}| \leq D_{vp+1}$ and the sequence $\{D_n\}_{n=0}^{\infty}$ is defined in (2.14). Otherwise, if $\Theta(vp,\beta) \neq 0$ for some $v = 1,2,\cdots$, then Eq.(1.8) has no analytic solutions in any neighborhood of the origin.

*Proof***.** We seek a power series solution of (1.8) of the form (2.1) as in the proof of Theorem 2.1, where the equality (2.4) or (2.5) is satisfied. If $\Theta(vp,\beta) \neq 0$ for some natura number v, then the equality in (2.15) does not hold for $n = vp$ since $1 - \beta^{vp} = 0$.. In such a circumstance Eq.(1.8) has no formal solutions.

When $\Theta(vp,\beta) = 0$ for all natural number v, for each v the corresponding b_{vp+1} in (2.15) has infinitely many choices in C, that is, the formal series solution (2.1) defines a family of solutions with infinitely many parameters. Choose $b_{vp+1} = \xi_{vp+1}$ arbitrarily such that

$$|\xi_{vp+1}| \leq D_{vp+1}, \qquad v = 1,2,\cdots, \tag{2.16}$$

Where D_{vp+1} is defined by (2.14). In what follows we prove that the formal series solution (2.1) converges in a neighborhood of the origin. Observe that $|\beta^n - 1|^{-1} \leq \Gamma$ for $n \neq vp$. It follows from (2.15) that

$$|b_{n+1}| \leq \Gamma L(\sum_{i=0}^{n-1}|b_{i+1}||b_{n-i}| + \sum_{i=0}^{n-1}\sum_{k=0}^{n-i}|b_{i+1}||b_k||b_{n-i-k}|), \tag{2.17}$$

for all $n \neq vp$, $v = 1,2,\cdots$.

Let

$$W(z,\mu,\eta,\Gamma,L) = \sum_{n=0}^{\infty} D_n z^n, \qquad D_0 = |\mu|, \qquad D_1 = |\eta|. \tag{2.18}$$

It is easy to check that (2.18) satisfies the implicit functional equation

$$R(z,\psi,\mu,\eta,\Gamma,L) = 0, \tag{2.19}$$

where R is defined in (2.8). Moreover, similarly to the proof of Theorem 2.1, we can prove that (2.19) has a unique analytic solution $\psi(z,\mu,\eta,\Gamma,L)$ in a neighborhood of the origin such that $\psi(0,\mu,\eta,\Gamma,L) = |\mu|$ and $\psi'_z(0,\mu,\eta,\Gamma,L) = |\eta|$ .Moreover, we also have $\psi(z,\mu,\eta,\Gamma,L) = W(z,\mu,\eta,\Gamma,L)$. Thus (2.18) converges in a neighborhood of the origin. Moreover, it is easy to show that, by induction,

$$|b_n| \le D_n, \qquad n = 1,2,\cdots.$$

Therefore, the series (2.1) converges in a neighborhood of the origin. This completes the proof.

ANALYTIC SOLUTION OF (1.2)

Theorem3.1. Suppose the conditions of Theorems 2.1 and 2.2 or Theorem 2.3 are satisfied and $b \neq 0$. Then Eq.(1.7) has an analytic solution of the form $y(z) = g(\beta g^{-1}(z))$ in a neighborhood of the number μ , where $g(z)$ is an analytic solution of (1.8).

Proof. By Theorem 2.1-2.3, we may find an analytic solution $g(z)$ of the auxiliary equation (1.8) in the form of (2.1) such that $g(0) = \mu$ and $g'(0) = \eta \neq 0$. Clearly the inverse $g^{-1}(z)$ exists and is analytic in a neighborhood of $g(0) = \mu$. If we define $y(z) = g(\beta g^{-1}(z))$, then

$$y'(z) = \beta g'(\beta g^{-1}(z))(g^{-1}(z))' = \frac{\beta g'(\beta g^{-1}(z))}{g'(g^{-1}(z))},$$

$$\begin{aligned}
&(a - y'(z))[y(y(z)) - ay(z)] \\
&= \frac{[ag'(g^{-1}(z)) - \beta g'(\beta g^{-1}(z))][g(\beta^2 g^{-1}(z)) - ag(\beta g^{-1}(z))]}{g'(g^{-1}(z))} \\
&= \frac{\beta[g(\beta g^{-1}(z)) - ag(g^{-1}(z))]^2 g'(\beta g^{-1}(z))}{g'(g^{-1}(z))} = [y(z) - az]^2 y'(z). \\
&= [y(y(z)) - ay(z)][y(z) - az]^2.
\end{aligned}$$

as required. This completes the proof.

We have shown that under the conditions of Theorem 2.1, 2.2 or 2.3, Eq.(1.7) has an analytic solution $y(z) = g(\beta g^{-1}(z))$ in a neighborhood of the number μ , where $g(z)$ is an analytic solution of (1.8).Since the function $g(z)$ in (2.1) can be determined by (2.4), it is possible to calculate, at least in theory, the explicit form of $y(z)$, an analytic solution of (1.2), in a neighborhood of the fixed point μ of $y(z)$ by means of (1.3). However, knowing that an analytic solution of (1.2) exists, we can take an alternative route as follows. Assume that $x(z)$ is of the form

$$x(z) = x(\mu) + x'(\mu)(z-\mu) + \frac{x''(\mu)}{2!}(z-\mu)^2 + \cdots; \tag{3.1}$$

we need to determine the derivatives $x^{(n)}(\mu)$, $n=0,1,2,\cdots$. First of all, in view of (1.6) and (1.3), we have

$$x(\mu)=\frac{b}{y(\mu)-a\mu}=\frac{b}{(1-a)\mu}.$$

and

$$x'(\mu)=\frac{b}{y(\mu)-a\mu}=\frac{b}{(1-a)\mu}.$$

Next by calculating the derivatives of both of (1.2), we obtain

$$x''(z)=x'(az+\frac{b}{x'(z)})[a-\frac{bx''(z)}{(x'(z))^2}].$$

Thus

$$x''(\mu)=\frac{ab}{(1-a)\mu[1+(1-a)\mu]}.$$

It seems from the above calculations that the higher derivatives $x^{(m)}(z)$ at $z=\mu$ can be determined uniquely in similar manners. Hence let $x^{(m)}(\mu)=\lambda_m$, it is then easy to write out the explicit form of our solution $x(z)$:

$$x(z)=\frac{b}{(1-a)\mu}+\frac{b}{(1-a)\mu}(z-\mu)+\frac{ab}{2!(1-a)\mu[(1+(1-a)\mu]}(z-\mu)^2+\sum_{n=3}^{\infty}\frac{\lambda_n}{n!}(z-\mu)^n.$$

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